FORM PROPORTIONALITIES IN THE PLANT DESIGN OF CASTEL DEL MONTE

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The seeming perfection of the geometric forms at Castel del Monte together with the singular choice of octagonal figures evince a design plan that many scholars have endeavored to decipher.

Most notable is the work of Heinz Götze, because of the historical and architectural context he provides, specifically as it relates to the architecture in the Mediterranean basin and the Muslim influence that Frederick II Hohenstaufen experienced growing up in Sicily.

Ancient designers relied on proportionalities for laying out architectural plans. These were judgmental proportions, appeals to ancient concepts of proportions such as the golden ratio, or ratios borrowed from the study of classic buildings. Understandably, scholars have presumed the existence of some proportionalities in the plant forms at Castel del Monte. The perfection of the repeating forms and surfaces connote a geometric order in the design.

Götze made a compelling case for proportionalities in the plant design of Castel del Monte. His work was echoed by Wulf Scirmer, whose study is the source of the plant measurement data. Götze focused on four octagons in the plant that he found to be related by singular proportions, Fig. 1. These are:

1. The courtyard octagon (“octagon C”)
2. The octagon formed at the indoor side of the façade wall (“octagon F”)
3. The plant octagon formed at the center point of the towers (“octagon D”), which is not a physical structure
4. The tower octagon (“octagon T”)

Götze concluded that octagon C is the reference in these proportional relationships. This likely resulted from his observation that octagon C matches very closely the center octagon in the eight-pointed star formed by connecting the center point of every third tower, Fig. 2. The segments of the eight-pointed star are referred to as the minor diagonals of the octagon in a new study, Fig. 3.

Götze concluded that these octagons have radial dimensions related as follow:

- Octagon F is in the ratio of 2:1 with octagon C
- Octagon D is in the ratio of ($\sqrt{2}$+1):1 with octagon C, or 2.4142:1
- Octagon T is in the ratio of ($\sqrt{2}$-1):1 with octagon C, or 0.4142:1

The actual ratios are close approximations when calculated from measurements. Götze came up with the exact mathematical ratios from a geometric analysis (Götze, pg. 165-169). He strived to demonstrate the geometric nature of the plant design by showing a number of mathematical relationships in the plant forms. He fitted the major forms of the plant (the courtyard octagon, the façade wall octagon and the tower octagons) into an exact geometric frame that gave the geometric proportions in the ground plan (Götze, pg. 168).

Götze devoted considerable discussions to these proportionality ratios to demonstrate the geometric foundation for the plant design. He showed how these proportions are verified by measurements (Götze, pg. 191-193), and highlighted similar proportions and figures from Islamic works in geometry.

Götze postulated a likely geometric construction process that produces the final plant layout with the identified proportion, Fig 4 [click here for animated figure - GIF format]. This procedure starts at the courtyard and moves progressively to define the façade wall and then the towers. The rooms, with their important cross vaults, are left out in this geometric design process, presumed to be defined in the space between the courtyard and the façade wall.

While correct in sustaining the geometric foundation of the plant design, Götze fell short of identifying the true geometric design process. His theorized geometric design replicates some of the major forms of the plant in the predicated proportions; but, it relies on approximations and does not provide a foundation for explaining key details, singularities and other irregularities.

His analysis, for example, consistently ignores the footing dimension that affects the position of the indoor side of the façade wall and the width of the façade wall. A major singularity are the tower spandrels that he brought to light. While he was successful at fitting the spandrels in the tower geometry, he could not explain their provenance, why the towers were designed misshapen, and how these all fit in the geometric design.

A shortcoming in Götze's works and those of other scholars is due consideration to the structural requirements for this edifice, especially with respect to cross vault construction.

Structural needs are important considerations, especially for a designer that is considering new architectural plans and has to appraise what is realizable structurally and what is not. This is especially true in the medieval mindset, when construction was not in the realm of the engineering discipline, but was an instinctive endeavor based on experience and shrouded in religious and mystical believes.

Castel del Monte is a unique medieval attempt that embraces a more scientific approach to the design, relying on geometry as the science to provide design answers. The structural needs of the cross vaults are combined with the distinctive features of the octagon to provide a unique structural arrangement that must have been perceived by the designers as a "heavenly revelation" to be embodied in the form of Castel del Monte.

A new theory has been developed for the plant design, discussed in details in other documents. This new plant design is a geometric algorithm that reproduces all basic plant forms in all details and with precise measures. This is itself evidence that the plant design is geometric in nature. The proportions that Götze sought to demonstrate the geometric nature of the plant design are all part of this new design model, because the geometric algorithm has a parallel mathematical algorithm that ties all plant forms mathematically.

This new design model embraces the cross vault structural requirements as the driving consideration for the plant design. The octagon with its minor diagonals offers a unique and elegant solution for cross vaults. The design, however, does not even starts with an octagon; it starts from a simple square. The geometric figures that next define the plant space and set the direction for the following geometric derivations are the two natural circles of this square: the circles inscribed and the circle circumscribing the square, Fig. 5.

A base octagon is inscribed within the outer circle and becomes the outset for laying out forms, Fig. 6. It is a surprising find not considered in previous studies, most likely because it is not a physical structure. The towers are defined first, nested in the corners of this base octagon. The geometric derivation progresses from the towers at the periphery, inward toward the center: façade wall, cross vault, and courtyard wall. The courtyard is the last items to be defined, much like a leftover space. This is the total opposite of the geometric design approach hypothesized by Götze.

The base octagon minor diagonals become the focus for the cross vaults and the resolution of their lateral thrusts, Fig. 7. The towers at the corners of the base octagon become terminations for the minor diagonal lines. The towers serve or were intended to counter the lateral forces emanating from the cross vaults.

There are eight rooms on a floor, each with a cross vault, which in turn creates four lateral thrusts along the diagonals of the cross vault square, for a total of 32 lateral thrusts flowing in different directions, Fig. 7. Amazingly, all these lateral thrusts coalesce along the minor diagonals (the eight-pointed star) when these cross vaults are located at the 90°-intersections of the minor diagonals. As a result, only eight piers are needed, one at each corner of the eight-pointed star, these are the eight towers.

This is the first phase of the design process, the one that defines the overall outline, the concept design, Fig. 8. There is a slight modification of the towers in a second phase of the design, the modified plant design, where many of the singularities and irregularities surface. The tower are enlarged by the measure of the footing width, f, and relocated slightly outward by a measure, i, that is a quantum smaller than the footing width. This caused a ripple of small adjustments at the façade wall, the cross vault and the courtyard wall; but the courtyard octagon remained unchanged.

It is revealing to revisit Götze's proportionalities among octagons C, D, F, and T from the perspective of the new design model. Fig. 9, 10, and 11 show an analysis, with related graphics, for a determination of Götze's theorized key proportions in this new geometric design model. No actual dimensions are needed, because the ratios are relationships among geometric forms, independent from their measures.

Table 1 shows a comparison of these proportional ratios between the new design model, panel 1 and 2, and the actual measures as discussed by Götze, panel 3.

Lines 2, 3, and 4 in Table 1 address the three key ratios highlighted by Götze. The sub-lines 2r, 3r, and 4r show the ratios theorized by Götze, which serve as reference for the comparison. Panel 1 shows the results of the concept design in the new model, while panel 2 shows the results after the modification at the towers. Panel 3 shows the ratios based on measurements documented by Götze and Schirmer.

Panel 1 shows an additional ratio on line 1, because of its significance; it is the ratio between the octagon formed by the inner-most corners of the towers (octagon A in Fig. 1) and the courtyard octagon. It is an octagon related to the façade wall, larger than octagon F by the footing measure (f in Fig. 8). Götze ignored the footing width, f; accordingly the two octagons F and A are the same in his analysis.

Panel 1 addresses the concept design, which is the idea, the pristine conception of the castle plant, before any modifications were made. Panel 1 shows that the ratios theorized by Götze are numerically exact in the concept design, with zero discrepancies, lines 1, 3 and 4.

Panel 3 shows the values of these proportional ratios based on the measurements reported by Götze and Schirmer. They are very close approximations of the theorized value. However, the discrepancies, very small for octagons F and D respectively on lines 2 and 3, cannot be dismissed as inaccuracies in the construction, which is extremely accurate for medieval standards of castle construction.

Panel 2 shows very similar deviations and it covers the modified plant design in the new model; it is a purely geometric design, with mathematical precision. The discrepancies from the theorized ratios in panels 2and 3 come from design modification at the towers, not from construction deviations.

The ratio for the tower octagon is a perfect ratio of (√2-1) in the concept design, line 4 in panel 1. This ratio shows a consistent discrepancy of over 5% in both the modified design of the new model, panel 2, and the actual measurements, panel 3. This is explained by the fact that the ideal ratio applies to the tower size in the concept design. The tower size was increased by the footing measure, f, in the second design phase, panel 2. The footing width is 5.2% of the original tower width and shows up in the calculation of the ratio, the 5% discrepancy .

Why then the other two ratios have a substantial lower discrepancies, lines 2 and 3? The ratio for the octagon F is almost perfect, differing from the theorized value by 1% or less. This is the result of a fortuitous coincidence. The perfect ratio of 2:1 applies to octagon A, line 1 of panel 1. Octagon F is smaller than octagon A by the footing measure, f; therefore the ratio is less than the perfect 2, by 2.1% as shown on line 2 of panel 1.

The towers were moved, spreading out from the center of the plant, in the second phase of the design. The outward relocation of the towers causes an enlargement of octagon F, because the façade wall is attached to the towers. The enlarged octagon F comes then very close to octagon A of the concept design; this explains the small discrepancy. The separation between octagon F and octagon A, the footing measure f, is diminished by the measure of the tower relocation i.

It seems it was a fortuitous coincidence focusing on the indoor side of the façade wall, octagon F, and ignoring the footing width, f. The real octagon in the ratio of 2:1 is octagon A very close to octagon F, the indoor side of the façade wall.

Table 1

Ratios		Panel 1			Panel 2			Panel 3		
		Concept Design			Modified Design			Measurements (Götze, Schirmer)		
	Octagon X : Courtyard octagon	Ratio of Variable	Ratio Value	% Diff	Ratio of Variable	Ratio Value	% Diff	Ratio of Variable	Ratio Value	% Diff
1	X = connecting tower inner corners	c_{13}/c_2	2.00	0.0%						
	Theorized ratio	A/C = 2	2.00							
2	X = at indoor side of façade wall	c_{12}/c_2	1.96	2.1%	c_{12}''/c_2	1.98	-1.0%	h_{12}''/h_2''	1.98	-0.9%
2r	Theorized ratio	F/C = 2	2.00		F/C = 2	2.00		F/C = 2	2.00	
3	X = connecting tower centers	c_{18}/c_2	2.41	0.0%	c_{18}''/c_2	2.44	1.3%	h_{18}''/h_2''	2.46	2.1%
3r	Theorized ratio	$D/C=(\sqrt{2}+1)$	2.41		$D/C=(\sqrt{2}+1)$	2.41		$D/C=(\sqrt{2}+1)$	2.41	
4	X= of tower	$(t/2)/c_2$	0.41	0.0%	$(t'/2)/c_2$	0.44	5.2%	$(u'/2)/h_2''$	0.44	5.3%
4r	Theorized ratio	$T/C=(\sqrt{2}-1)$	0.41		$T/C=(\sqrt{2}-1)$	0.41		$T/C=(\sqrt{2}-1)$	0.41	

Notes:

1. A = octagon formed by connecting the eight inner-most corner of the tower corners
2. C = courtyard octagon
3. D = octagon formed by connecting the center points of the towers
4. F = octagon formed by indoor side of the façade wall
5. T = tower octagon
6. Ratios are specified by variables names used in the study of the new design model, Fig. 12.
7. Panels 1 and 2 use measurements along the corner direction of the plant octagon; panel 3 uses measurements along the wing (side) direction of the plant octagon because the latter are the complete measurement set reported by Götze and Schirmer. The two set of measurements are related by a constant (cos 22.5°), which cancels out in the ratio calculation.

Castel del Monte is unique among medieval castles. The medieval designers embraced a different approach for the basic design, more scientific from our perspective. They used geometry as the design science; they used geometry not to just to give forms to structures but to define the forms, their location, their size, and how they are interconnected.

They let geometry define all forms in a continuous geometric derivation process that starts from a simple square, a true algorithm. There is no need for a drawing of the plan layout and measures. All that is needed is the knowledge of the algorithm, and the plan is deployed every time starting with a square of the preset dimension and following the geometric algorithm. This is likely how the plant design was laid out in the field to trace the foundations and start construction.

Götze, H. ed. 1998. *Castel del Monte, Geometric Marvel of the Middle Ages*. New York: Prestel-Verlag.

Schirmer, W. ed. 2001. *Castel del Monte, Forschungsergebnisse der Jahre 1990 bis 1996.* Verlag Philipp Von Zabern, Mainz. [*Castel del Monte, Research results from 1990 to 1996.* Publisher Philipp Von Zabern, Mainz.]v

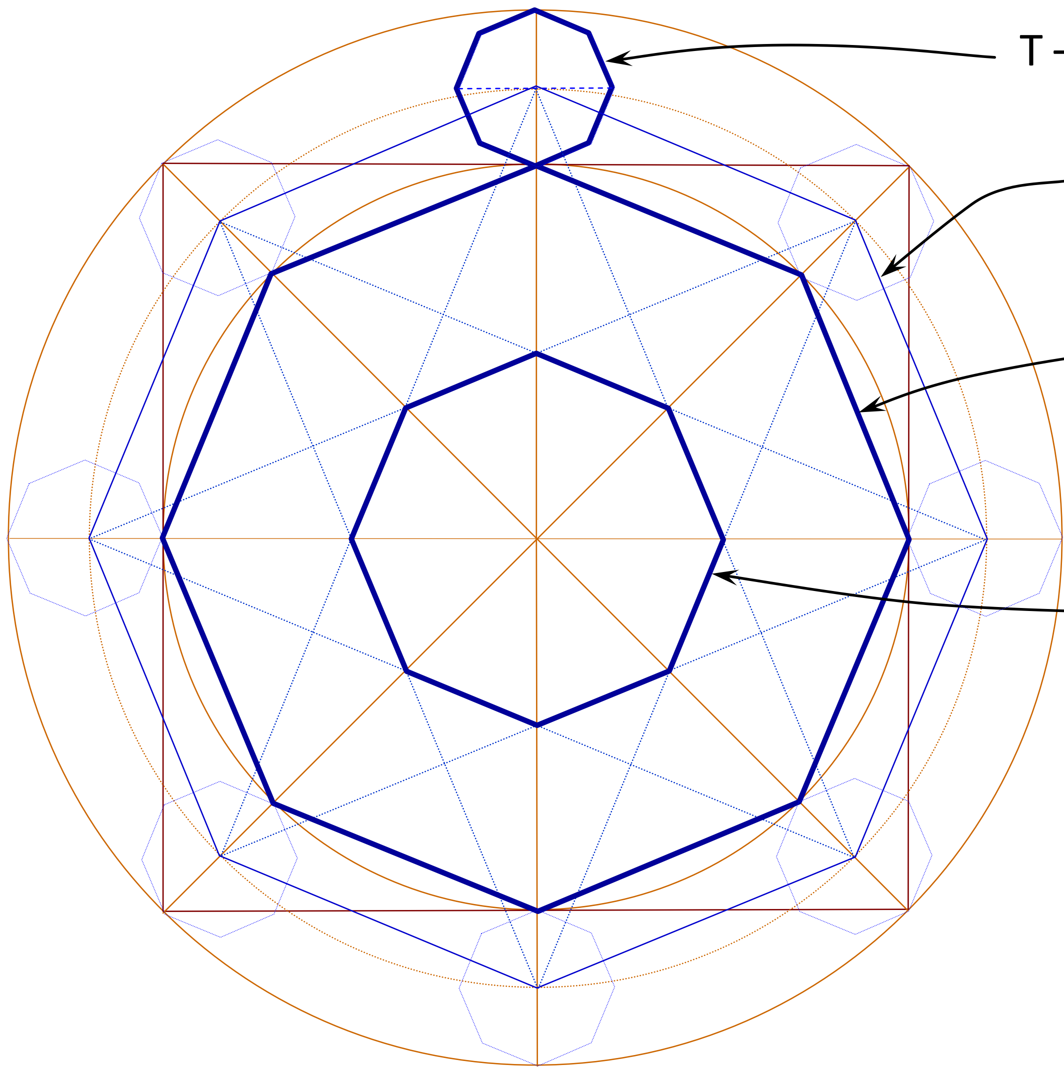


Fig. 1. Octagons Involved in H. Götze ’s Proportionalities

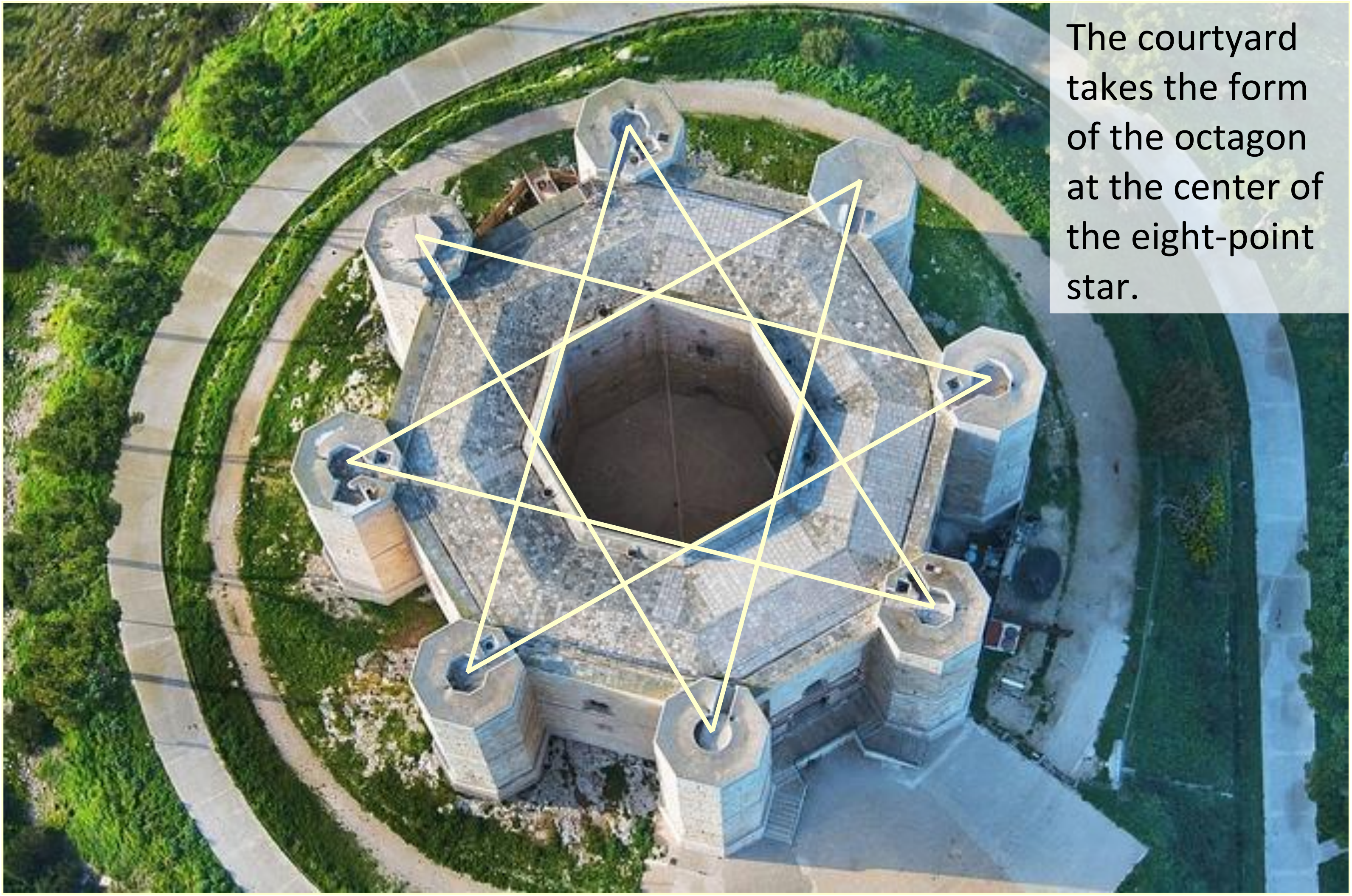


Fig. 2. Courtyard at the Center of the Islamic Star Formed by the Towers

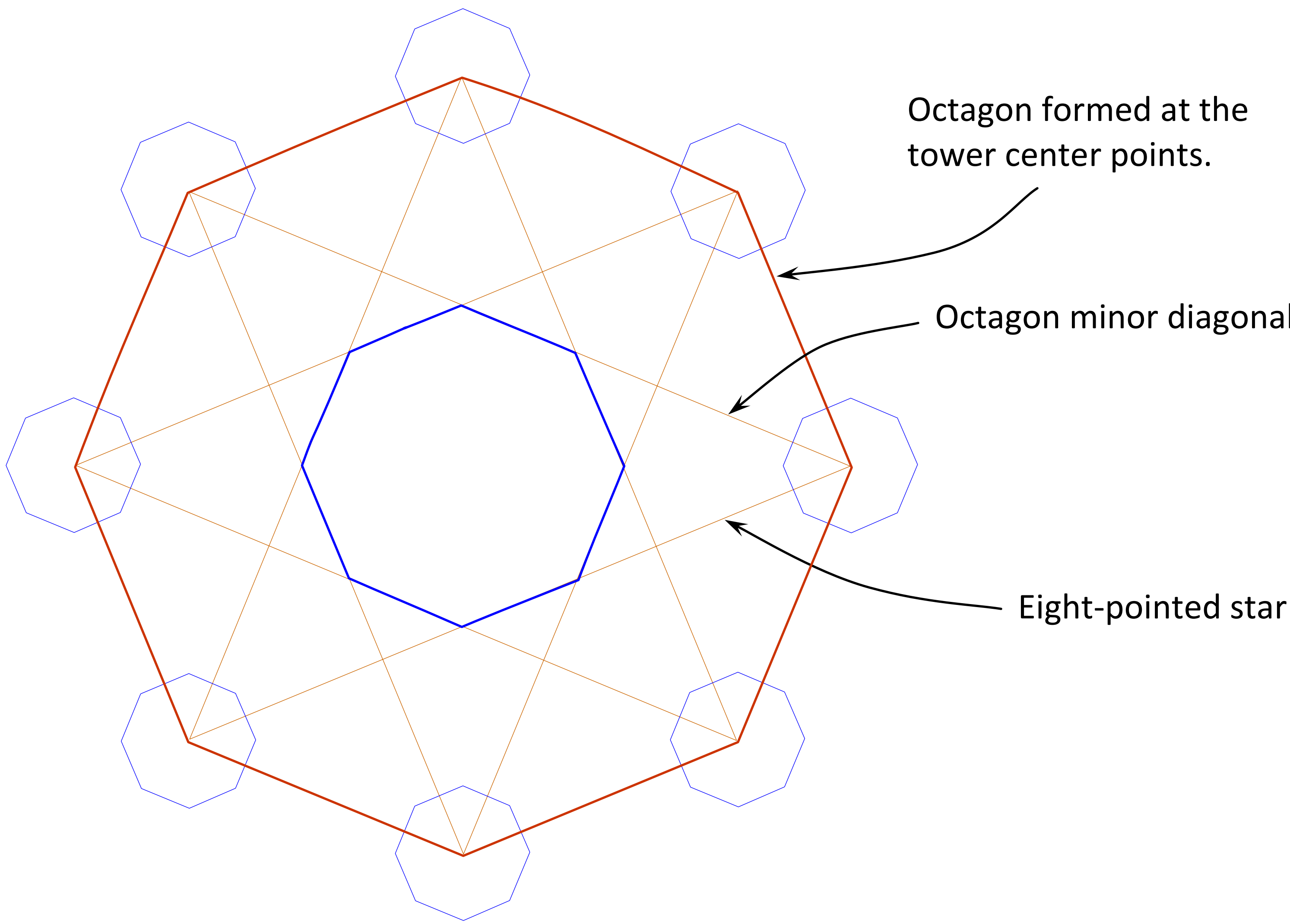


Fig. 3. Octagon at the Tower Centers and the Minor Diagonals

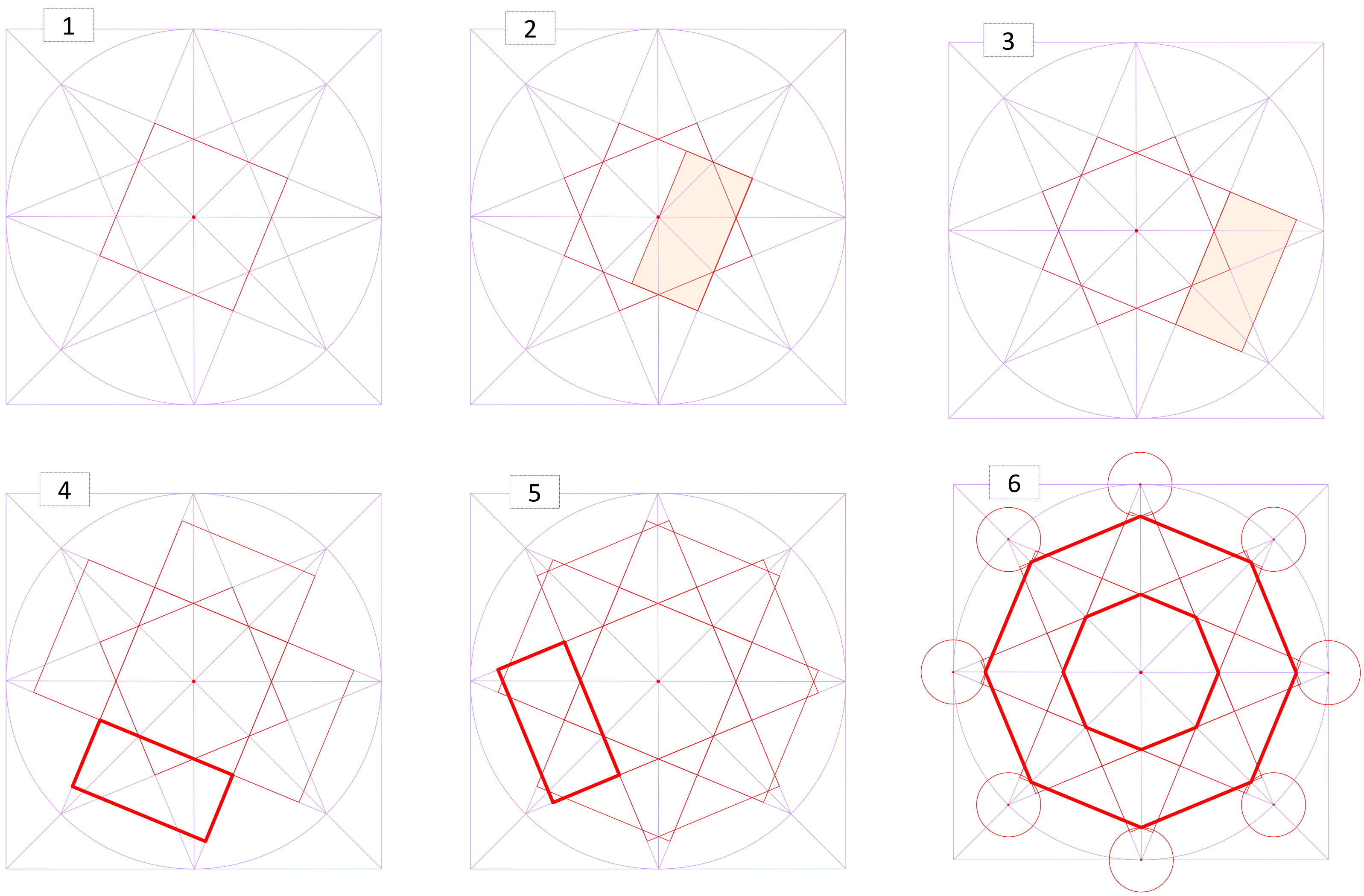


Fig. 4. Selected Steps in H. GÖtze's Geometric Plan Design

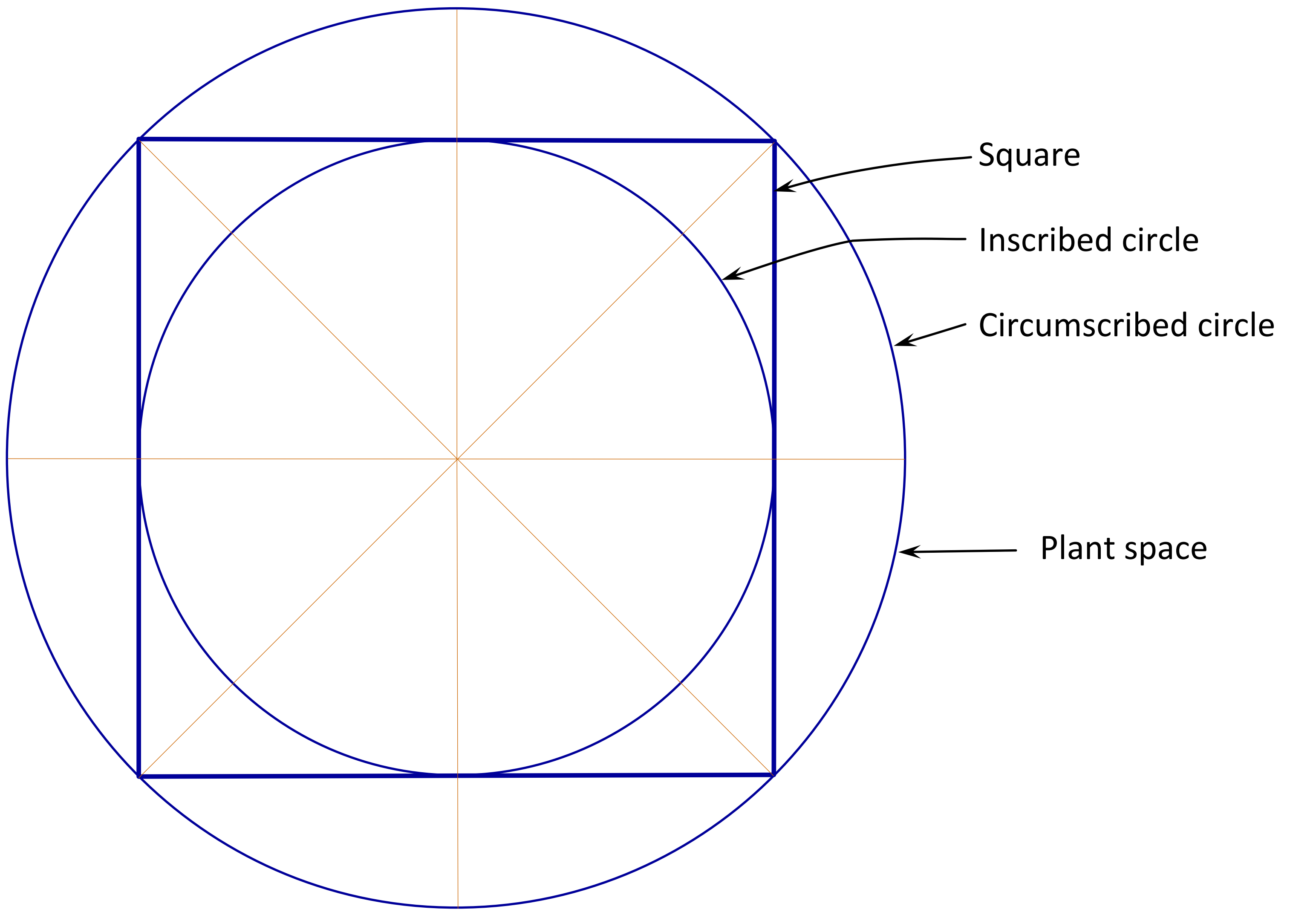


Fig. 5. Starting Geometric Figures in the Plant Design

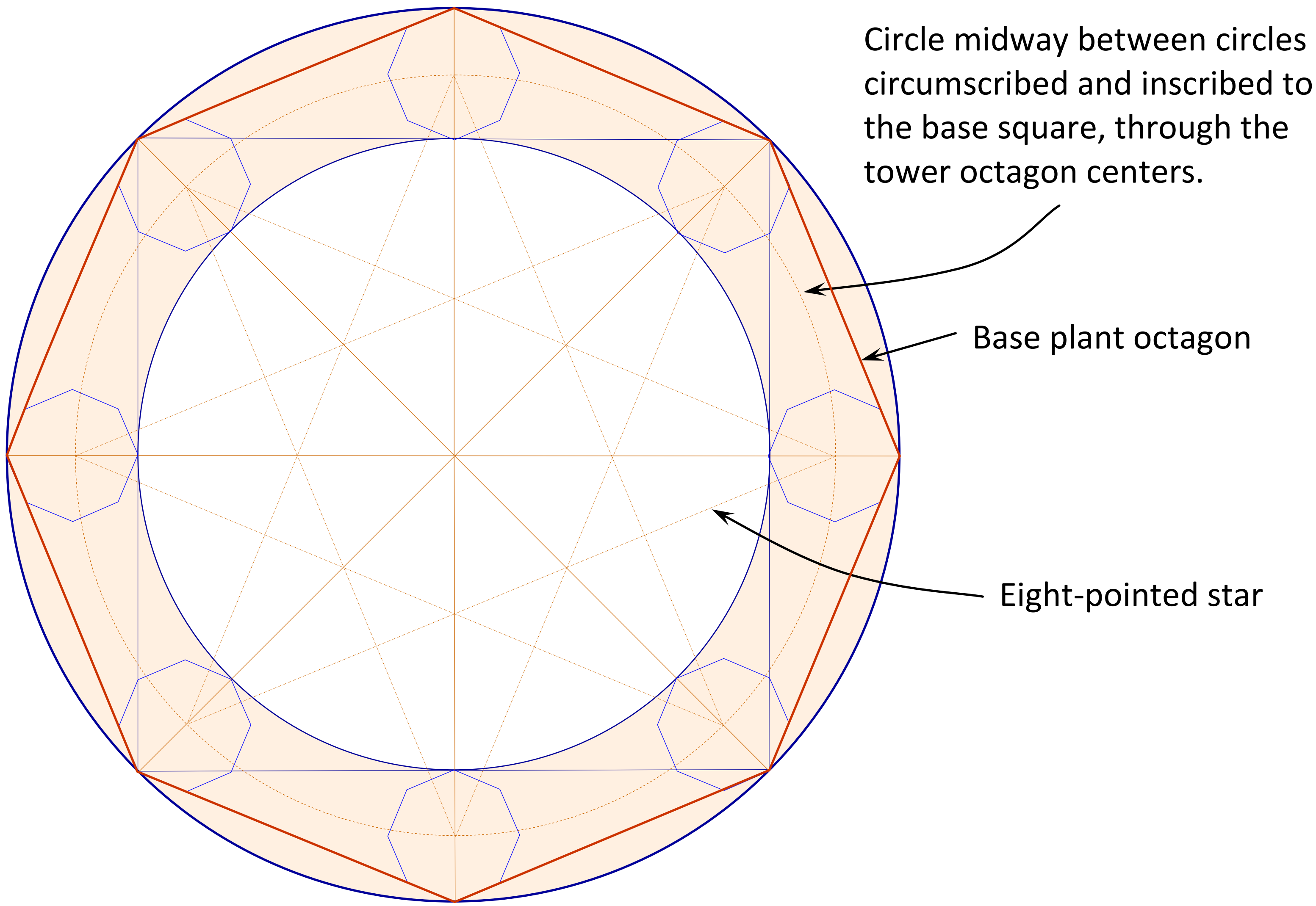


Fig. 6. Base Octagon and Towers in the Tower Space

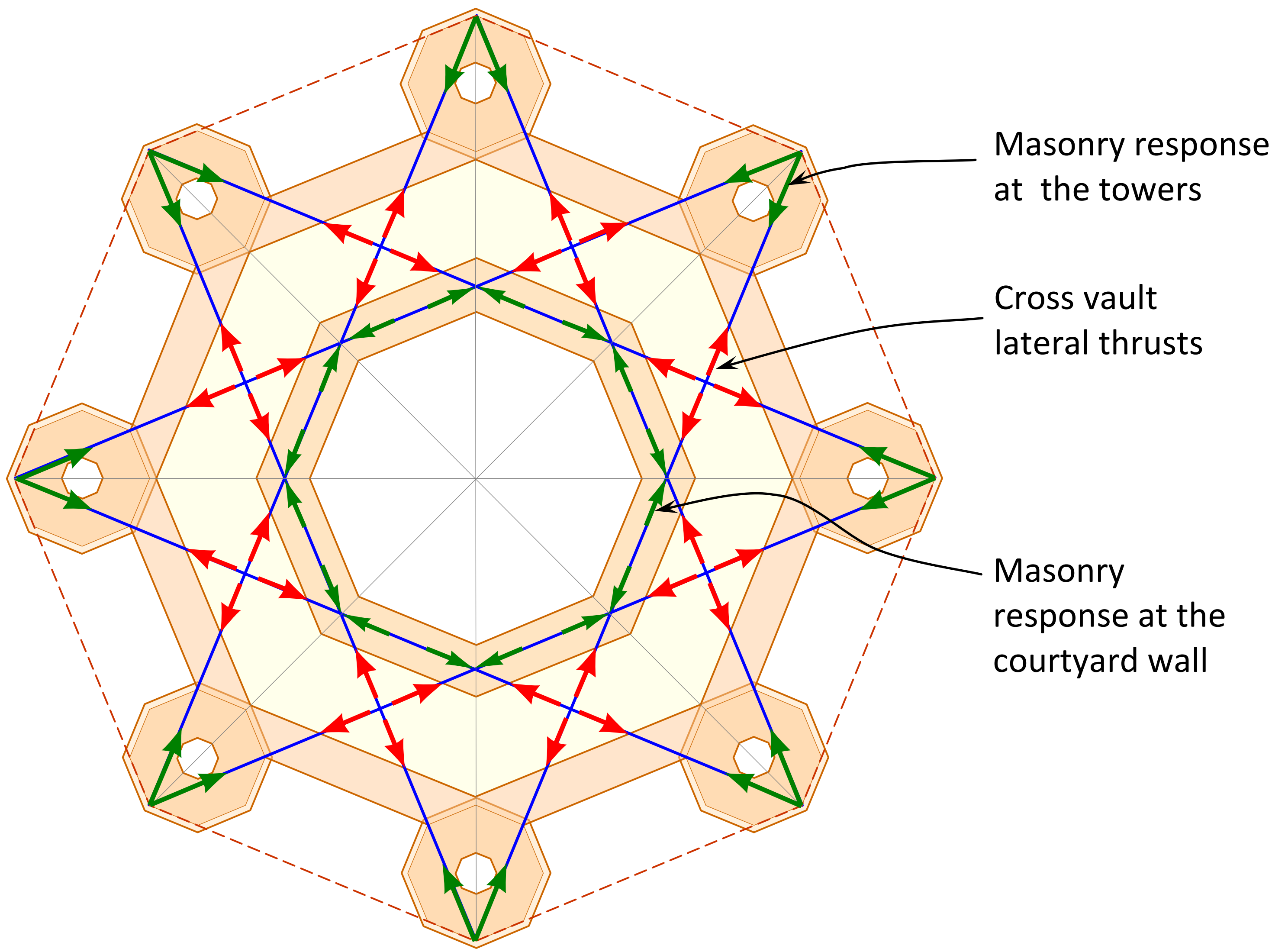


Fig. 7. Cross Vault Lateral Thrust Solution at Castel del Monte

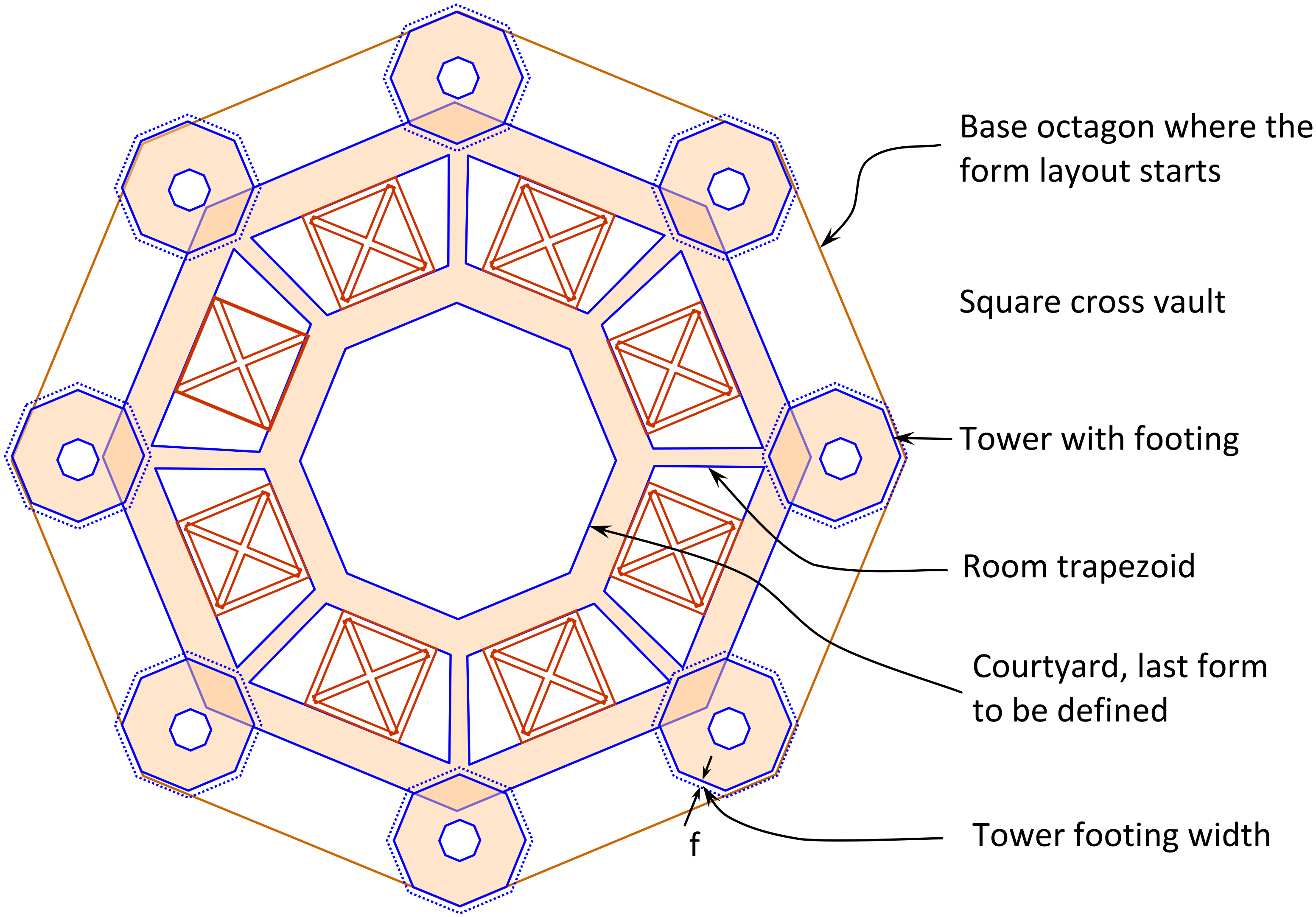
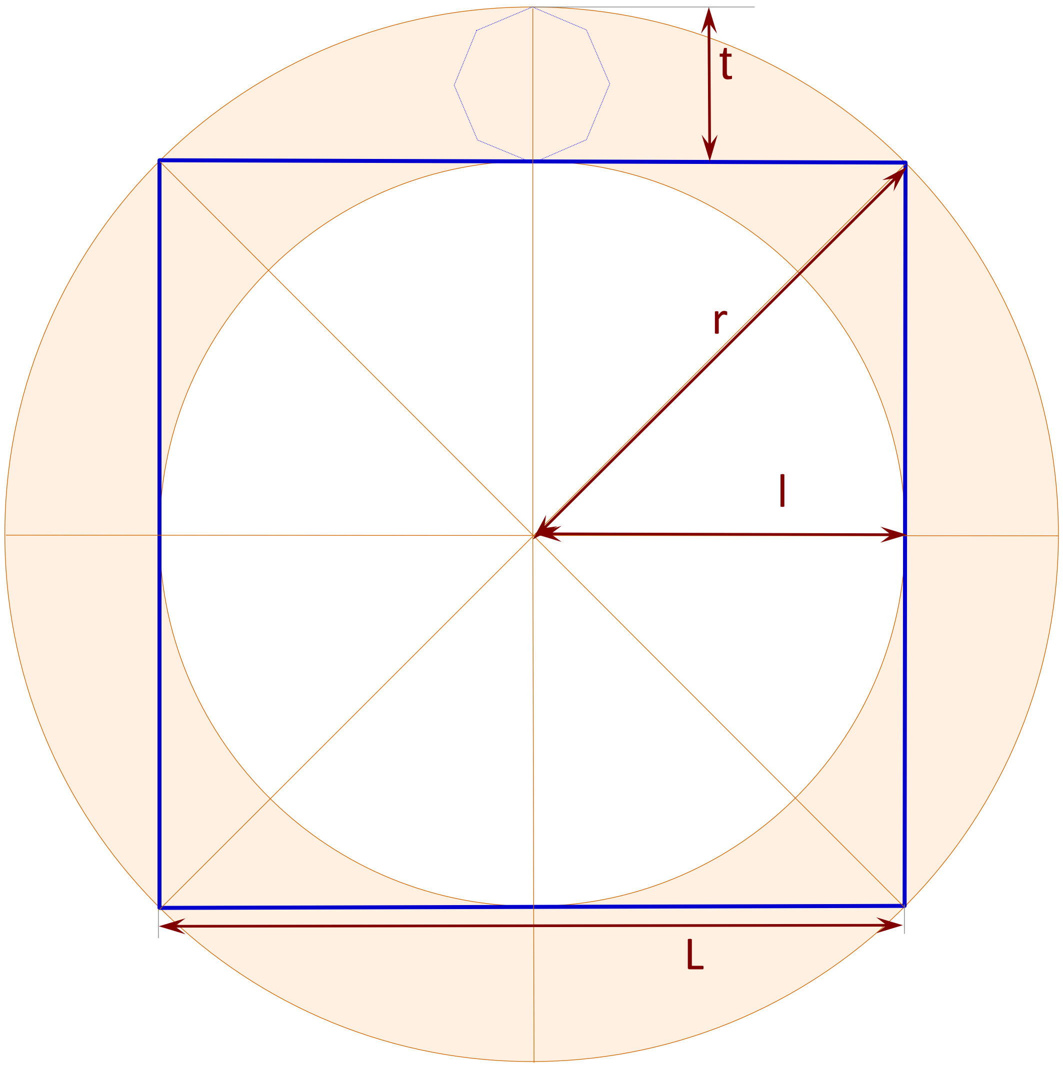
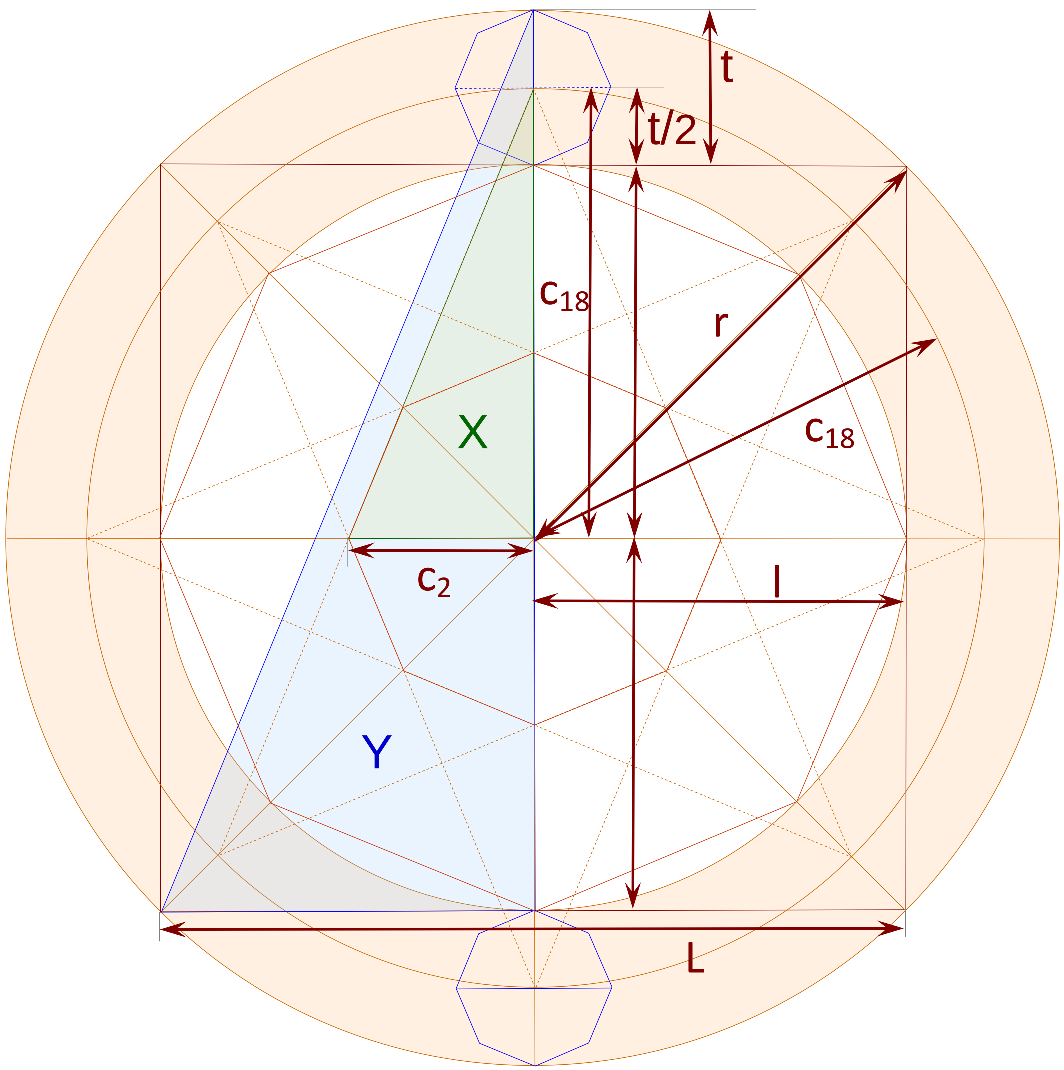


Fig. 8. Basic Plant Layout per the New Geometric Design Plan



$$l = L / 2$$
$$r = l \cdot \sqrt{2}$$
$$t = r - l = l \cdot (\sqrt{2}-1)$$

Fig. 9. Basic Measures Related to the Base Square



All dimensions are easily determined geometrically, and can be defined as functions of l.

$r = l \cdot \sqrt{2}$

$t = r - l = l \cdot (\sqrt{2}-1)$

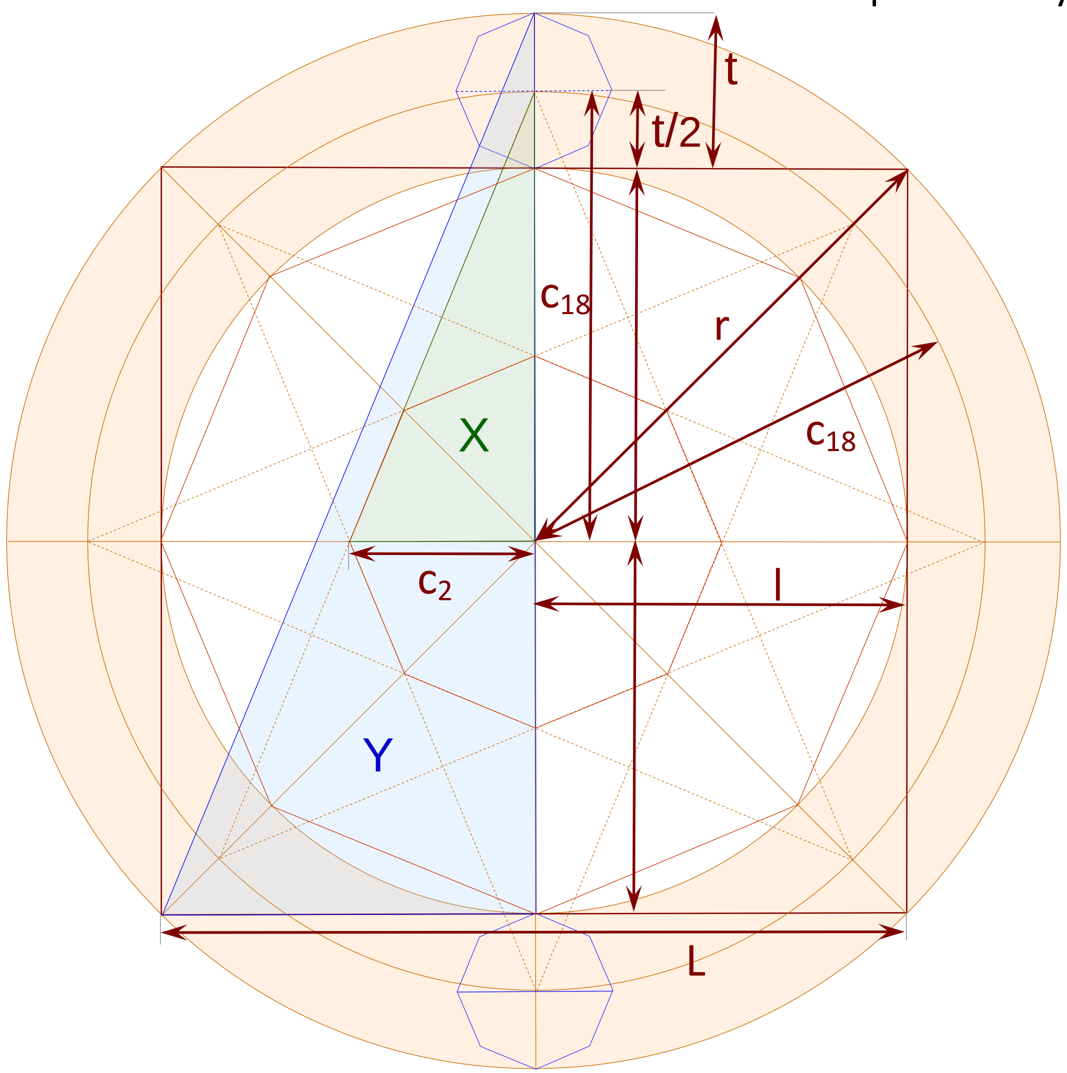
$t/2 = (l / 2) \cdot (\sqrt{2}-1)$

$c_{18} = l + t / 2 = (l / 2) \cdot (\sqrt{2}+1)$

c_2 is not readily determined mathematically.

Triangles X and Y are similar and serve to determine the dimension of c_2.

Fig. 10. Geometric Forms For the Analysis of Proportionalities



Proportionality ratio determination:

$l = L / 2$
$r = l \cdot \sqrt{2}$
$t = r - l = l \cdot (\sqrt{2}-1)$
$t/2 = (l / 2) \cdot (\sqrt{2}-1)$
$c_{18} = l + t / 2 = (l / 2) \cdot (\sqrt{2}+1)$

X / Y proportionality
$c_2 / c_{18} = l / (2 \cdot l + t)$
$c_2 = (l \cdot c_{18}) / (2 \cdot l + t) = l / 2$

H. GÖtze's proportionalities defined geometrically:
$c_2 : l : c_{18} = c_2 : 2 \cdot c_2 : (\sqrt{2}+1) \cdot c_2$
$c_2 : l : c_{18} = 1 : 2 : (\sqrt{2}+1)$
$t/2 = (l/2) \cdot (\sqrt{2}-1) = c_2 \cdot (\sqrt{2}-1)$

Fig. 11. H. Götze's Proportionality Ratios

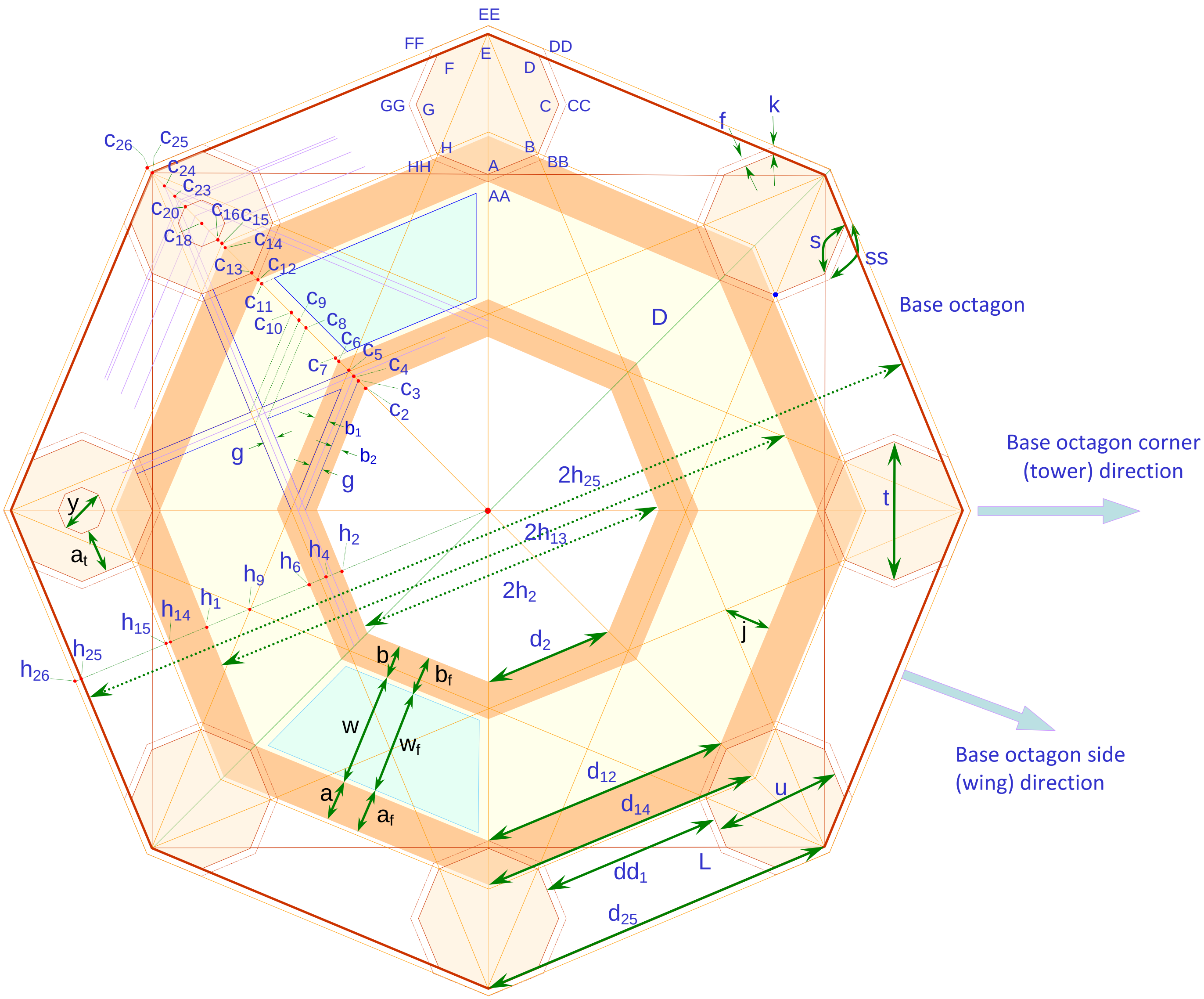


Fig. 9. Graphical References for the Mathermathical Variable